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MEAN TOPOLOGICAL DIMENSION

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ABSTRACT

In this paper we present some results and applications of a new invariant for dynamical systems that can be viewed as a dynamical analogue of topological dimension. This invariant has been introduced by M. Gromov, and enables one to assign a meaningful quantity to dynamical systems of infinite topological dimension and entropy. We also develop an alternative approach that is metric dependent and is intimately related to topological entropy.

1. Introduction

One of the basic invariants of a dynamical system (X, T) is its topological entropy. This quantifies to what extent nearby points diverge as the system evolves. For the shift on $\{1, 2, \ldots, k\}^{\mathbb{Z}}$, the topological entropy is $\log k$ and thus gives a dynamical interpretation of the cardinality of the set of states. For the shift $K^{\mathbb{Z}}$, where K is an infinite compact space, this invariant is always $+\infty$, and thus gives no information about K other than the fact that it is infinite. Recently, M. Gromov suggested a definition of a new dynamical invariant, the mean dimension, that would recover in the above example, for a nice K, its topological dimension. Our purpose in this paper is to explore several ways of defining the mean dimension and to relate this new invariant to some other problems in topological dynamics.

A classical theorem of Beboutov states that any real flow (X, T_t) whose fixed point set can be imbedded in $\mathbb{R}$ can be imbedded in the space of continuous functions on $\mathbb{R}$, with the natural action of $\mathbb{R}$ (see Kakutani [5]). An open problem for many years (see Auslander [1] and [4]) has been to decide whether every

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minimal system (X,T) is imbeddable in $([0,1]^{\mathbb{Z}},\sigma)$, where σ is the shift operation. As we will see, a necessary condition for such an imbedding is that the mean dimension of (X,T) be at most 1. Since it is not hard to see that there are minimal systems with any value for the mean dimension, it follows that, in general, the $\mathbb{Z}$-analogue of Beboutov's theorem fails. In a sequel to this paper by the first author (Lindenstrauss [7]), it will be shown that, conversely, any extension of a minimal system, with mean dimension at most Cd (where C is some constant), can be imbedded in the coordinate shift on the Hilbert cube $([0,1]^d)^{\mathbb{Z}}$.

The initial examples encountered in topological dynamics of systems with a unique invariant measure were equicontinuous systems and their isometric extensions. These examples fostered the belief that these systems had special metric properties. This belief was dispelled with the discovery of Jewett that any weakly mixing system has a uniquely ergodic model. Since then, there have been no results indicating that uniquely ergodic systems have special dynamical properties. We will relate zero mean dimension to a dynamical version of being totally disconnected, and it will then turn out that any uniquely ergodic system has mean dimension zero.

In the usual approach to topological dimension, one focuses on the degree to which open covers by sets of small diameter overlap without paying any attention to the cardinality of these covers. However, as Pontryagin and Schnirelmann [10] have shown, one can also define the topological dimension using only cardinalities of open covers. This suggests that the mean dimension should be directly related to the topological entropy of a system, and indeed we shall establish such a relationship which will imply, in particular, that any system with finite topological entropy has zero mean dimension.

While we have written this paper for the usual system (X,T) (i.e. a $\mathbb{Z}$-action), all of the results given here can be established for discrete amenable groups, in particular $\mathbb{Z}^d$. The reader familiar with this extension will have no difficulty in extending all proofs and definitions to the amenable case, with the exception of a useful subadditivity lemma for which we include a complete proof. This lemma apparently has been known by ergodic theorists, but has not been published before.

OVERVIEW: In §2 we review some well known facts related to dimension theory, most of which can be found in [3], define the mean dimension and deduce some simple properties of it. In §3 we compute the mean dimension for some examples, and indicate how to construct minimal systems with an arbitrary value for the mean dimension. In §4 we define an analogue of Minkowski dimension, the metric mean dimension, which is very closely related to the topological entropy, and investigate the connection between this and the mean dimension. In §5 we discuss the small boundary property (the dynamical version of being dynamically totally disconnected), and show that a system that has this property has mean dimension zero. Finally, in the appendix we prove the result alluded to above concerning subadditive functions defined on finite subsets of amenable groups. This last result is all that is needed for the knowledgeable reader to extend the results and definitions we present to actions of discrete amenable groups.

This paper is part of the first-named author's PhD thesis, prepared under the guidance of the second-named author.

2. Preliminaries and basic properties of mean dimension

Let X be a compact metric space, α a finite open cover of X. We will say that a cover β refines α ($\beta \succ \alpha$), if every member of β is a subset of some member of α.

Definition 2.1: If α is an open cover of X we shall denote

$$\operatorname{ord}(\alpha) = \max_{x \in X} \sum_{U \in \alpha} 1_U(x) - 1 \qquad \text{and} \qquad \mathcal{D}(\alpha) = \min_{\beta \succ \alpha} \operatorname{ord}(\beta),$$

where β runs over all finite open covers of X refining α.

$\mathcal{D}(\alpha)$ has a very important property which we shall use heavily — it is subadditive, i.e. $\mathcal{D}(\alpha \vee \beta) \leq \mathcal{D}(\alpha) + \mathcal{D}(\beta)$ (for open covers α and β of X, their join $\alpha \vee \beta$ is defined to be the open cover whose elements are $U \cap V$ for all $U \in \alpha$, $V \in \beta$). To verify this we shall give another characterization of $\mathcal{D}(\alpha)$ using dimension. We recall the definition of the (cover) dimension for compact metric spaces: X has dimension $\leq D$ if for every open cover α of X there is a refinement $\beta \succ \alpha$ with $\operatorname{ord}(\beta) \leq D$ — i.e. if for every open cover α, $\mathcal{D}(\alpha) \leq D$.

Definition 2.2: A continuous map $f\colon X \to Y$ will be called α-compatible if it is possible to find a finite open cover of $f(X)$, β, such that $f^{-1}(\beta) \succ \alpha$. We will use the notation $f \succ \alpha$ to denote that f is α-compatible.

Proposition 2.3: *If X is compact, $f\colon X \to Y$ a continuous function such that for every $y \in Y$, $f^{-1}(y)$ is a subset of some $U \in \alpha$, then f is α-compatible.*

Proof: Without loss of generality we assume $f(X) = Y$. If $\alpha = \{U_1, \ldots, U_n\}$, we define for $i = 1, \ldots, n$,

$$V_i = \{y \in Y\colon f^{-1}(y) \subset U_i\}.$$

By our assumption, $\bigcup_{i=1}^n V_i = Y$, and clearly $f^{-1}(V_i) \subset U_i$. Thus, it only remains to show the V_i's are open. Assume to the contrary that there is some $y \in V_i$ such that there is a sequence $y_n \to y$ with $y_n \notin V_i$, i.e.

$$f^{-1}(y_n) \cap (X \setminus U_i) \neq \emptyset.$$

Let $x_n \in f^{-1}(y_n) \cap (X \setminus U_i)$. The x_n have a subsequence that converge to some $x \in X$. As f is continuous,

$$f(x) = \lim f(x_{n_k}) = \lim y_{n_k} = y$$

and so $x \in f^{-1}(y) \subset U_i$. On the other hand, for every n, $x_n \in X \setminus U_i$, which implies that

$$x = \lim x_{n_k} \in X \setminus U_i,$$

a contradiction. ∎

We can now state the following characterization of $\mathcal{D}(\alpha)$:

PROPOSITION 2.4: *If α is an open cover of X, then*

$$\mathcal{D}(\alpha) \leq k$$

iff there is an α-compatible continuous function f: $X \to K$ where K has topological dimension k.

Proof: Suppose there exists such an f: $X \to K$, and a cover β of $f(X) \subset K$ such that $f^{-1}(\beta) \succ \alpha$. By k-dimensionality of K, and thus of $f(X)$, the cover β has a subcover γ with $\mathrm{ord}(\gamma) \leq k$. And so

$$f^{-1}(\gamma) \succ f^{-1}(\beta) \succ \alpha.$$

As $\mathrm{ord}(f^{-1}(\gamma)) = \mathrm{ord}(\gamma) \leq k$ we are done.

Conversely, suppose $\gamma \succ \alpha$, with $\mathrm{ord}\,\gamma \leq k$. We shall construct a γ compatible map from X to a k-dimensional simplicial complex $|\mathcal{C}|$.

Indeed, let $\{w_U\}_{U \in \gamma}$ be a partition of unity subordinate to γ, i.e. functions such that for any $x \in X$

$$\sum_{U \in \gamma} w_U(x) = 1,$$

and $\mathrm{supp}(w_U) \subset U$. The vertices of $\mathcal{C}$ correspond to the elements of γ, the s-dimensional simplices to all families $\{U_0, \ldots, U_s\}$ with $\bigcap_{i=0}^s U_i \neq \emptyset$. The points in each simplex $\{U_0, \ldots, U_s\}$ can be parameterized as

$$\sum_{i=0}^s a_i[U_i], \quad a_i \geq 0 \text{ for every } i, \text{ and } \sum_{i=0}^s a_i = 1.$$

We now map every point x of X into the simplex $\{[U]: x \in U \in \gamma\}$ of $|\mathcal{C}|$ by

$$x \mapsto \sum_{x \in U \in \gamma} w_U(x)[U].$$

This is clearly continuous. f is γ compatible (and hence α compatible) since for every $y \in |\mathcal{C}|$, $f^{-1}(y)$ is a subset of every U that corresponds to a vertex of the least dimensional simplex of $|\mathcal{C}|$ containing y. ∎

COROLLARY 2.5: *$\mathcal{D}$ is sub-additive, i.e. if α and β are finite open covers of X*

$$\mathcal{D}(\alpha \vee \beta) \leq \mathcal{D}(\alpha) + \mathcal{D}(\beta).$$

Proof: If $f: X \to A$ is α-compatible, with $\dim A = \mathcal{D}(\alpha)$, and $g: X \to B$ is β-compatible, with $\dim B = \mathcal{D}(\beta)$, then $h: X \to A \times B$ with $h: x \mapsto (f(x), g(x))$ is $\alpha \vee \beta$ compatible. By the well known properties of dimension, $\dim A \times B \leq \dim A + \dim B$, and hence we have an $\alpha \vee \beta$-compatible mapping from X to a space of dimension $\leq \mathcal{D}(\alpha) + \mathcal{D}(\beta)$. ∎

If α is an open cover of X, we will use the notation α_a^b for $b > a \in \mathbb{Z}$ to denote the open cover

$$\alpha_a^b = T^{-a}\alpha \vee T^{-a-1}\alpha \vee \cdots \vee T^{-b}\alpha.$$

Definition 2.6: If (X, T) is a dynamical system, then the mean dimension of (X, T), denoted by $\operatorname{mdim}(X, T)$ (or $\operatorname{mdim}(X)$ if T is understood), is defined by

$$\operatorname{mdim}(X, T) = \sup_{\alpha} \lim_{n \to \infty} \frac{\mathcal{D}(\alpha_0^{n-1})}{n},$$

where α runs over all finite open covers of X.

Remarks:

- The fact that the limit above exists is a consequence of the sub-additivity of $\mathcal{D}$, and is true also for the general case of amenable group actions (see appendix). In this case, the limit over n is replaced by taking the limit on subsets of the group that become more and more invariant.
- It is clear from the definition that if X' is a T-invariant closed subset of X then

$$\operatorname{mdim} X' \leq \operatorname{mdim} X.$$

However, unlike topological entropy, mdim of a factor may be greater than the mdim of the original system. The same is, of course, true for the usual topological dimension.

- Recall that the topological dimension can be defined by

$$\dim X = \sup_\alpha \mathcal{D}(\alpha)$$

where α runs over all open covers of X. It follows that if X has finite topological dimension, then for any α and n

$$\mathcal{D}(\alpha_0^{n-1}) < \dim X,$$

and hence $\operatorname{mdim}(X) = 0$.
- In the definition of mdim one can replace $\sup_\alpha$ by

$$\sup_k \lim_{n\to\infty} \frac{\mathcal{D}(\alpha(k)_0^{n-1})}{n}$$

where $\alpha(k)$ is any sequence of open covers such that the maximal diameter of an element of $\alpha(k)$ tends to zero as $k \longrightarrow \infty$.

We now prove some easy results about mdim, showing that this new invariant is well behaved.

PROPOSITION 2.7: *For any dynamical system (X, T),*

$$\operatorname{mdim}(X, T^n) = n \cdot \operatorname{mdim}(X, T).$$

Proof: Let α be any open cover of X. Then

$$\begin{aligned}\lim_{k\to\infty} \frac{\mathcal{D}(\alpha \vee T^{-n}\alpha \vee \cdots \vee T^{-(k-1)n}\alpha)}{k} &\le n \cdot \lim_{k\to\infty} \frac{\mathcal{D}(\alpha_0^{kn-1})}{kn} \\ &\le n \operatorname{mdim}(X, T)\end{aligned}$$

and so $\operatorname{mdim}(X, T^n) \le n \cdot \operatorname{mdim}(X, T)$. On the other hand,

$$\alpha_0^{kn-1} = \alpha_0^{n-1} \vee T^{-n}\alpha_0^{n-1} \vee \cdots \vee T^{-(k-1)n}\alpha_0^{n-1}$$

and so

$$\operatorname{mdim}(X, T) = \sup_\alpha \lim_{k\to\infty} \frac{\mathcal{D}(\alpha_0^{kn-1})}{kn} \le \frac{\operatorname{mdim}(X, T^n)}{n}. \qquad \blacksquare$$

PROPOSITION 2.8: *Let (X_i, T_i) be a sequence of dynamical systems, for $1 \le i < R$, $R \in \mathbb{N} \cup \{\infty\}$. Then*

$$\operatorname{mdim}(X_1 \times X_2 \times \cdots, T_1 \times T_2 \times \cdots) \le \sum_{i<R} \operatorname{mdim}(X_i, T_i).$$

Proof: Let α be any open cover of $X_1 \times X_2 \times \cdots$. There is an $N < R$ and open covers $\beta(j)$ of X_j such that

$$\beta = \bigvee_{j=1}^{N} \pi_j^{-1}(\beta(j)) \succ \alpha,$$

where π_j is the projection onto the jth coordinate. By subadditivity of $\mathcal{D}(\cdot)$,

$$\mathcal{D}(\beta_0^n) \leq \sum_{j=1}^{N} \mathcal{D}(\beta(j)_0^n)$$

and so

$$\lim_{n\to\infty} \frac{\mathcal{D}(\beta_0^{n-1})}{n} \leq \sum_{j=1}^{N} \lim_{n\to\infty} \frac{\mathcal{D}(\beta(j)_0^{n-1})}{n} \leq \sum_{j=1}^{N} \mathrm{mdim}(X_j). \qquad \blacksquare$$

3. Some examples

We begin with calculating the mean dimension of some natural dynamical systems.

PROPOSITION 3.1: *Let K be finite dimensional and compact, and let σ be the shift on $K^{\mathbb{Z}}$. Then*

$$\mathrm{mdim}(K^{\mathbb{Z}}, \sigma) \leq \dim K.$$

Proof: Let α be any open cover of $K^{\mathbb{Z}}$. For every $a < b$, let $\pi_a^b\colon K^{\mathbb{Z}} \to K^{b-a+1}$ be the projection of an element of $K^{\mathbb{Z}}$ to its coordinates with indices in the range $a \dots b$. As the topology on $K^{\mathbb{Z}}$ is the Tychonoff topology, there is a refinement α' of α, and an N such that every $U \in \alpha'$ is of the form $\left(\pi_{-N}^{N}\right)^{-1}(V)$ for some open $V \subset K^{2N+1}$.

Thus to show $\mathrm{mdim}(K^{\mathbb{Z}}) \leq \dim K$ it suffices to show that for α' of the above form,

$$\lim_{n\to\infty} \frac{\mathcal{D}(\alpha_0'^{n-1})}{n} \leq \dim K.$$

However, $\alpha_0'^{n-1}$ is actually the inverse image (via π_{-N}^{N+n-1}) of a cover of K^{2N+1+n}. Since K^{2N+1+n} has dimension at most $(2N+1+n)\dim K$, there is a cover $\tilde{\beta}$ that refines $\pi_{-N}^{N+n-1}(\alpha')$ with $\mathrm{ord}(\tilde{\beta}) \leq (2N+1+n)\dim K$. As

$$\left(\pi_{-N}^{N+n-1}\right)^{-1}(\tilde{\beta}) \succ \alpha_0'^{n-1}$$

we see that

$$\frac{\mathcal{D}(\alpha'^{n-1}_0)}{n} \leq \frac{(2N+1+n)\dim K}{n} \longrightarrow \dim K \quad \text{as } n \longrightarrow \infty. \qquad \blacksquare$$

There are compact subsets K of $\mathbb{R}^d$ such that $\dim K \times K < 2\dim K$, and so the above result cannot be improved in general. However, for nice K we have indeed that $\operatorname{mdim} K^{\mathbb{Z}} = \dim K$. We will need the following dimension theoretic lemma:

LEMMA 3.2: *If α is an open cover of $[0,1]^d$ such that no $U \in \alpha$ intersects two opposing faces of the cube, then* $\operatorname{ord}(\alpha) \geq d$.

Proof: Like most results of this type, this is merely a reformulation of the Brouwer Fixed Point Theorem. Indeed, suppose $\operatorname{ord}(\alpha) < d$. We can use this to construct a continuous map F from $[0,1]^d$ to $|\mathcal{C}|$, a $d-1$ (or lower) dimensional simplicial complex with a vertex $[U]$ corresponding to each $U \in \alpha$ and simplices corresponding to all families $\{U_0, \ldots, U_s\}$ with $\bigcap_{i=0}^s U_i \neq \emptyset$ (just like we did in the proof of Proposition 2.4).

Now define G: $|\mathcal{C}| \to [0,1]^d$ as follows: first map the vertices of $\mathcal{C}$ by G: $[U] \mapsto (e_0(U), \ldots, e_{d-1}(U))$ where

$$e_k(U) = \begin{cases} 1 & \text{if } U \text{ intersects the face } x_k = 0 \text{ of } [0,1]^d, \\ 0 & \text{otherwise,} \end{cases}$$

then extend this to the simplices of $|\mathcal{C}|$ in an affine way. $G \circ F$: $[0,1]^d \to [0,1]^d$ maps the d dimensional cube to a finite number of at most $d-1$ dimensional simplices, so its image cannot contain $(0,1)^d$. Let

$$p \in (0,1)^d \setminus \operatorname{Image}(G \circ F),$$

and take H to be the projection that sends an $x \in [0,1]^d \setminus \{p\}$ along the ray emanating from p and passing through x to the boundary of $[0,1]^d$. The map $H \circ G \circ F$ maps the d dimensional cube to its boundary, and maps each face of the cube to the opposite face, so this map has no fixed points neither inside the cube nor on its boundary, in contradiction to the Brouwer Fixed Point Theorem. $\blacksquare$

We now prove that if $K = [0,1]^d$ then $\operatorname{mdim} K^{\mathbb{Z}} = \dim K$. With an eye to a construction of a minimal system that is not imbeddable in $([0,1]^{\mathbb{Z}}, \sigma)$ we prove a slightly more general statement. If $I = \{\cdots i_0 < i_1 < i_2 \cdots\}$ is a finite or infinite subset of $\mathbb{Z}$, we take π_I to be the projection

$$\pi_I\colon (\ldots, x_0, x_1, \ldots) \to (\ldots, x_{i_0}, x_{i_1}, \ldots)$$

(π_I is a function from $K^{\mathbb{Z}}$ to $K^{|I|}$, $K^{\mathbb{N}}$ or $K^{\mathbb{Z}}$ depending on I).

PROPOSITION 3.3: *Let X be a shift invariant closed subset of $([0,1]^d)^{\mathbb{Z}}$ such that there is an infinite set of indices $I \subset \mathbb{N}$ and an $\bar{x} \in X$ such that*

1. *I has upper density θ, i.e.*
$$\theta = \overline{\lim_{n\to\infty}} \frac{|I \cap \{0, \ldots, n-1\}|}{n};$$
2. *any $x \in ([0,1]^d)^{\mathbb{Z}}$ with*
$$\pi_{\mathbb{Z}\setminus I}(x) = \pi_{\mathbb{Z}\setminus I}(\bar{x})$$
is in X;

then

$$\mathrm{mdim}(X) \geq \theta d.$$

In particular, $\mathrm{mdim}\left(([0,1]^d)^{\mathbb{Z}}, \sigma\right) = d$.

Proof: Let α' be a cover of $[0,1]^d$ such that no element of α' intersects two opposing faces of the cube $[0,1]^d$, and let $\alpha = \pi_1^{-1}(\alpha')$. α is an open cover of $([0,1]^d)^{\mathbb{Z}}$, and since $X \subset ([0,1]^d)^{\mathbb{Z}}$ it induces an open cover $\tilde{\alpha}$ of X.

We now estimate $\mathcal{D}(\tilde{\alpha}_0^{n-1})$. Let $I(n) = I \cap \{0, \ldots, n-1\}$. For any $\beta \succ \mathcal{D}(\tilde{\alpha}_0^{n-1})$ the collection of sets

$$\beta' = \left\{\pi_{I(n)}\left(\{x \in U\colon \pi_{\mathbb{Z}\setminus I}(x) = \pi_{\mathbb{Z}\setminus I}(\bar{x})\}\right) : U \in \beta\right\}$$

is an open cover of $([0,1]^d)^{|I(n)|}$, and clearly no element of β' can intersect two opposing faces of the cube $([0,1]^d)^{|I(n)|}$. Thus $\mathrm{ord}(\beta') \geq d|I(n)|$, and this also means that $\mathrm{ord}(\beta) \geq d|I(n)|$. Since β was an arbitrary refinement of $\tilde{\alpha}_0^{n-1}$ we see that $\mathcal{D}(\tilde{\alpha}_0^{n-1}) \geq d|I(n)|$, hence

$$\mathrm{mdim}(X) \geq \lim_{n\to\infty} \frac{\mathcal{D}(\tilde{\alpha}_0^{n-1})}{n} \geq \overline{\lim_{n\to\infty}} \frac{d|I(n)|}{n} = \theta d. \qquad \blacksquare$$

Remark: In a very similar way, one can see that if $X \subset ([0,1]^{\mathbb{N}})^{\mathbb{Z}}$, and for a set $I \subset \mathbb{N}$ of indices with positive upper density and $\bar{x} \in X$ we have that any x with $\pi_{\mathbb{Z}\setminus I}(x) = \pi_{\mathbb{Z}\setminus I}(\bar{x})$ is in X, then $\mathrm{mdim}(X) = \infty$.

Since mean dimension is a topological invariant of (X, T), and the mean dimension of a closed invariant subset of a dynamical system is at most the mean dimension of the full system, we get:

COROLLARY 3.4: *A necessary condition for (X,T) to be imbeddable in the dynamical system $\big(([0,1]^d)^{\mathbb{Z}},\sigma\big)$ is that* $\operatorname{mdim} X \leq d$.

The standard methods of constructing minimal subshifts yield the existence of minimal systems with mean dimension r for all $r \in R^+ \cup \{\infty\}$. Any such example with mean dimension greater than 1 will not be imbeddable in $([0,1]^Z,\sigma)$; any example with infinite mean dimension will not be imbeddable in any $(([0,1]^d)^{\mathbb{Z}},\sigma)$. We remark that it is very easy to show that any dynamical system can be imbedded in $(([0,1]^{\mathbb{N}})^{\mathbb{Z}},\sigma)$ — indeed, any compact metric set X can be imbedded in $[0,1]^{\mathbb{N}}$ (say using the map ϕ), and thus

$$\Phi\colon x \longmapsto (\dots,\phi(T^{-1}x),\phi(x),\phi(Tx),\dots)$$

is an imbedding of (X,T) in $(([0,1]^{\mathbb{N}})^{\mathbb{Z}},\sigma)$. For completeness, we show in detail how to construct a minimal shift invariant closed subset of $([0,1]^2)^{\mathbb{Z}}$ with $\operatorname{mdim}(X) > 1$.

PROPOSITION 3.5: *There is a shift invariant closed subset X of $([0,1]^2)^{\mathbb{Z}}$ such that (X,σ) is minimal and* $\operatorname{mdim}(X) > 1$.

Proof: We consider $([0,1]^2)^{\mathbb{Z}}$ as bi-infinite words from the (rather large) alphabet $D = [0,1]^2$. As a metric on this space we take

$$d(x,y) = \sum_{i=-\infty}^{\infty} 2^{-i}\|x-y\|$$

where $\|\cdot\|$ is any norm on $\mathbb{R}^2$ (for example, the Euclidian norm). It will be convenient to use for $a<b$ the notation

$$x_a^b = (x_a, x_{a+1},\dots,x_b)$$

(we will use this notation also for $x \in D^N$ for finite N; in this case we begin to index the coordinates of x from 0).

We will build a decreasing sequence X_n of closed shift invariant subsets of $D^{\mathbb{Z}}$, and a decreasing sequence of subsets $I_n \subset \mathbb{N}$ such that:

1. The orbit of every point $x \in X_n$ comes within distance $C2^{-n}$ of every $y \in X_n$.
2. There is an $\bar{x}(n) \in X_n$ such that any $y \in D^{\mathbb{Z}}$ with $\pi_{\mathbb{Z}\smallsetminus I_n}(\bar{x}(n)) = \pi_{\mathbb{Z}\smallsetminus I_n}(y)$ is also in X_n.
3. $\pi_{\mathbb{Z}\smallsetminus I_n}(\bar{x}(m)) = \pi_{\mathbb{Z}\smallsetminus I_n}(\bar{x}(n))$ for any $m \geq n$.
4. $I = \bigcap_n I_n$ has upper density greater than 0.9.

From (1), $X = \bigcap_n X_n$ is minimal. From (2) and (3), if $\bar{x}$ is any limit point of the set $\{\bar{x}(n)\}$ then for all n, any y with $\pi_{\mathbb{Z}\setminus I_n}(\bar{x}) = \pi_{\mathbb{Z}\setminus I_n}(y)$ is in X_n, hence any y with $\pi_{\mathbb{Z}\setminus I}(\bar{x}) = \pi_{\mathbb{Z}\setminus I}(y)$ is in X. Finally, from (4) and Proposition 3.3,

$$\operatorname{mdim}(X) \geq 1.8.$$

So, it remains to construct the X_n, $\bar{x}(n)$ and I_n. The X_n will be of block type: there will be an $L_n \in \mathbb{N}$ and $B_n \subset D^{L_n}$ such that

$$X_n = \left\{(\dots x_{-1}, x_0, x_1, \dots)\colon \exists k \in \mathbb{Z} \text{ s.t. } \forall a \in \mathbb{Z},\ x_{k+aL_n}^{k+aL_n+L_n-1} \in B_n\right\}.$$

Such a (nonempty) system of block type is always topologically transitive, i.e. there is an $\tilde{x} \in X_n$ such that its orbit is dense in X_n. We take $\bar{x}(n)$ to be any point of X_n such that

$$x_{aL_n}^{aL_n+L_n-1} \in B_n \quad \text{for all } a \in \mathbb{Z}.$$

We start from $L_1 = 1$, $B_1 = D$, $X_1 = D^{\mathbb{Z}}$, and $I_1 = \mathbb{N}$. We proceed by induction; so assume we have defined L_n, B_n and I_n. Take $\tilde{x}$ to a point in X_n with dense orbit. By replacing $\tilde{x}$ by its shift if necessary, we can assume $x_{aL_n}^{aL_n+L_n-1} \in B_n$ for all $a \in \mathbb{Z}$. We take l_{n+1} to be an integer divisible by $2L_n$ such that for any $y \in X_n$ there is a

$$k \in \{-l_{n+1} + 10n, \dots, l_{n+1} - 10n\}$$

such that $d(y, \sigma^k \tilde{x}) < 2^{-n}$. We define $L_{n+1} = 2^{n+10} l_{n+1}$, and B_{n+1} by

$$B_{n+1} = \{b \in D^{L_{n+1}}\colon \forall a,\ b_{aL_n}^{aL_n+L_n-1} \in B_n \text{ and } b_{L_{n+1}-2l_{n+1}}^{L_{n+1}-1} = \tilde{x}_{-l_{n+1}}^{l_{n+1}-1}\}.$$

Finally, we define

$$I_{n+1} = I_n \cap \{k \equiv 0, \dots, L_{n+1} - 2l_{n+1} - 1 \pmod{L_{n+1}}\}.$$

We leave to readers the verification that X_n, $\bar{x}(n)$ and I_n so defined have the required properties. ∎

We have already calculated the mean dimension of $[0,1]^{\mathbb{Z}}$; now we wish to consider factors of this system. In Lindenstrauss [6] a relatively simple argument was used to show that all factors of $[0,1]^{\mathbb{Z}}$ have infinite topological entropy. We use a similar argument to show that all the factors of $[0,1]^{\mathbb{Z}}$ have strictly positive mean dimension.

THEOREM 3.6: *If (Y,S) is any nontrivial factor of $([0,1]^{\mathbb{Z}},\sigma)$ then*

$$\mathrm{mdim}(Y) > 0.$$

Proof: Let ϕ: $[0,1]^{\mathbb{Z}} \to Y$ be the factor map from $([0,1]^{\mathbb{Z}},\sigma)$ to (Y,S). Let $\alpha = \{U_0, U_1\}$ be an open cover of Y such that neither U_0 nor U_1 is dense in Y. We shall prove that

$$\lim_{n\to\infty} \frac{\mathcal{D}(\alpha_0^{n-1})}{n} > 0.$$

Indeed, for any β that refines α_0^{n-1}, $\phi^{-1}(\beta)$ refines $\phi^{-1}(\alpha_0^{n-1})$ and

$$\mathrm{ord}(\phi^{-1}(\beta)) = \mathrm{ord}(\beta).$$

Thus it suffices to prove that for any open cover $\tilde{\alpha} = \{V_1, V_2\}$ of $[0,1]^{\mathbb{Z}}$, such that for $i = 0, 1$, $\mathrm{cl}\, V_i \neq [0,1]^{\mathbb{Z}}$,

$$\lim_{n\to\infty} \frac{\mathcal{D}(\tilde{\alpha}_0^{n-1})}{n} > 0. \tag{3.1}$$

As the V_i's are not dense, there are $p_i \in V_i$, N, and ϵ such that if $\|\pi_{-N}^N x - \pi_{-N}^N p_i\|_\infty < \epsilon$ then $x \notin V_{1-i}$. By replacing V_i with

$$\{x \in [0,1]^{\mathbb{Z}}\colon \|\pi_{-N}^N x - \pi_{-N}^N p_{1-i}\|_\infty > \epsilon\}$$

if necessary, we can assume the V_i are cylindrical sets that depend only on the coordinates $-N \ldots N$. Find some line $\ell \subset [0,1]^{2N+1}$ such that $\ell \not\subset \mathrm{cl}(\pi_{-N}^N V_i)$ for $i = 0$ and 1. Limiting our attention to the subset

$$X' = \{x \in [0,1]^{\mathbb{Z}}\colon \forall k\ \pi_{-N+(2N+1)k}^{N+(2N+1)k}(x) \in \ell\}$$

we can use similar arguments as in the proof that $\mathrm{mdim}([0,1]^{\mathbb{Z}}) = 1$ to see that

$$\mathcal{D}(\hat{\alpha} \vee \sigma^{-(2N+1)}\hat{\alpha} \vee \cdots \vee \sigma^{-(k-1)(2N+1)}\hat{\alpha}) = k,$$

where $\hat{\alpha}$ is the restriction of the cover $\tilde{\alpha}$ to X'. It follows that for all k

$$\frac{\mathcal{D}(\tilde{\alpha}_0^{k(2N+1)-1})}{k(2N+1)} \geq \frac{1}{2N+1},$$

and thus $\mathrm{mdim}(Y) \geq \frac{1}{2N+1} > 0$. ∎

It is also possible (with some effort) to find a minimal dynamical system with all its factors having strictly positive mean dimension — the example in Lindenstrauss [6] of a minimal system with no finite entropy factors also has no zero mean dimensional factors. A similar example can be found in Glasner and Maon [2].

We conclude this section with a simple example of a system with zero topological dimension (and hence also zero mean dimension) with $[0,1]^{\mathbb{Z}}$ as a factor: Let C be Cantor's set. C can be mapped (say by the map ϕ) onto $[0,1]$. Then $(\ldots, x_0, x_1, \ldots) \mapsto (\ldots, \phi(x_0), \phi(x_1), \ldots)$ is a factor mapping $(C^{\mathbb{Z}}, \sigma) \longrightarrow ([0,1]^{\mathbb{Z}}, \sigma)$. This example shows that, unlike entropy, factor maps can definitely increase mean dimension.

4. The metric mean dimension

In this section we present another definition of mean dimension, and investigate the connection between this new mean dimension, which we will call the metric mean dimension, and $\mathrm{mdim}(X)$. In particular, we will learn something new about $\mathrm{mdim}(X)$ — if $h_{\mathrm{top}}(X)$ is finite, $\mathrm{mdim}(X) = 0$. The metric mean dimension depends on the metric d, and can be thought as a mean Minkowski dimension.

For an open cover α, define the mesh of α according to the metric d by

$$\mathrm{mesh}(\alpha, d) = \max_{U \in \alpha} \mathrm{diam}(U).$$

Definition 4.1: Let X be a dynamical system, $d(\cdot,\cdot)$ a metric on X. Define

$$d_N(x,y) = \max_{0 \le n < N} d(T^n x, T^n y).$$

Set

$$S(X,\epsilon,d) = \lim_{n\to\infty} \inf_{\mathrm{mesh}(\alpha,d_n)<\epsilon} \frac{\log|\alpha|}{n}. \tag{4.1}$$

S is monotone nondecreasing as $\epsilon \to 0$, and we wish to measure just how fast it increases. We define the **metric mean dimension** of X (for the given metric d), $\mathrm{mdim_M}(X,d)$, as

$$\mathrm{mdim_M}(X,d) = \underline{\lim}_{\epsilon\to 0} \frac{S(X,\epsilon,d)}{|\log\epsilon|}. \tag{4.2}$$

To get a topological invariant we shall set

$$\mathrm{mdim_M}(X) = \inf_{d \text{ metric on } X} \mathrm{mdim_M}(X,d).$$

Notice that if $\mathrm{mesh}(\alpha, d_n) < \epsilon$ and $\mathrm{mesh}(\beta, d_m) < \epsilon$ then

$$\mathrm{mesh}(\alpha \vee T^{-n}\beta, d_{n+m}) < \epsilon.$$

Thus $\inf_{\mathrm{mesh}(\alpha,d_n)<\epsilon} \log|\alpha|$ is a subadditive function of n, and the limit in (4.1) can be replaced by an infimum.

Recall how the topological entropy is defined for a dynamical system X. Let X be a dynamical system, $d(\cdot,\cdot)$ a metric on X. A set $S \subset X$ is called (n,ϵ,d)-spanning if for every $x \in X$ there is a $y \in S$ so that for all $0 \leq k < n$,

$$d(T^k x, T^k y) < \epsilon.$$

Set

$$A(X,n,\epsilon,d) = \min\{|S| \colon S \subset X \text{ is } (n,\epsilon,d)\text{-spanning}\}.$$

The topological entropy is defined in terms of these as

$$\begin{aligned} S'(X,\epsilon,d) &= \overline{\lim_{n\to\infty}} \frac{\log A(X,n,\epsilon,d)}{n}; \\ h_{\mathrm{top}}(X) &= \lim_{\epsilon\to 0} S'(X,\epsilon,d). \end{aligned} \tag{4.3}$$

Notice that

$$S'(X,\epsilon,d) \geq S(X,2\epsilon,d) \geq S'(X,2\epsilon,d).$$

Indeed, the first inequality is a consequence of the fact that if S is (n,ϵ,d)-spanning, the d_n balls of radius ϵ cover X — and these are $|S|$ sets of d_n diameter smaller than 2ϵ. The second inequality is a consequence of the fact that if α is a cover with $\mathrm{mesh}(\alpha,d_n) < \epsilon$, and we choose one point from each element of α to form a set S, then S is (n,ϵ,d) spanning. Thus (4.2) is equivalent to

$$\mathrm{mdim_M}\,(X,d) = \underline{\lim_{\epsilon\to 0}} \frac{S'(X,\epsilon,d)}{|\log\epsilon|}.$$

And, in particular, the metric mean dimension will be nonzero only if $h_{\mathrm{top}}(X) = \infty$.

The rest of this section is dedicated to showing that the mean dimension of X is not larger than the metric mean dimension for all metrics d on X.

THEOREM 4.2: *For any dynamical system (X,T), and any metric d compatible with the topology on X,*

$$\mathrm{mdim}(X) \leq \mathrm{mdim_M}\,(X,d).$$

As $\mathrm{mdim}(X) = \sup_\alpha \lim \mathcal{D}(\alpha_0^{n-1})/n$, it is enough to prove that for any finite open cover α of X,

$$\lim_{n\to\infty} \frac{\mathcal{D}(\alpha_0^{n-1})}{n} \leq \mathrm{mdim_M}\,(X,d).$$

We can refine α to be of the form

$$\alpha = \{U_1, V_1\} \vee \{U_2, V_2\} \vee \cdots \vee \{U_r, V_r\},$$

where for every i, $\{U_i, V_i\}$ is a two element open cover of X.

Set $w_i \colon X \to [0,1]$ by

$$w_i(x) = \frac{d(x, X \setminus V_i)}{d(x, X \setminus V_i) + d(x, X \setminus U_i)}.$$

Notice that w_i is Lipschitz, and

$$\begin{aligned} U_i &= w_i^{-1}[0,1), \\ V_i &= w_i^{-1}(0,1]. \end{aligned} \tag{4.4}$$

Let C_L be a bound on the Lipschitz constant of all w_i. For any N define $F(N, \cdot) \colon X \to [0,1]^{rN}$ by

$$\begin{aligned} F(N,x) = \big(&w_1(x), w_2(x), \ldots, w_r(x), \\ &w_1(Tx), w_2(Tx), \ldots, w_r(Tx), \\ &\vdots \\ &w_1(T^N x), w_2(T^N x), \ldots, w_r(T^N x)\big). \end{aligned}$$

From (4.4) we see that $F(N, \cdot) \succ \alpha_0^{N-1}$. As usual, if $S \subset \{1, \ldots, rN\}$, then $F(N,x)_S \in [0,1]^{|S|}$ is the projection of $F(N,x)$ to the coordinates in the index set S. We need the following two lemmas:

LEMMA 4.3: *Let $\epsilon > 0$, $D = \mathrm{mdim}_{\mathrm{M}}(X,d)$. If N is larger than some $N(\epsilon)$, there is a $\xi \in (0,1)^{rN}$ such that for any $|S| \geq (D+\epsilon)N$,*

$$\xi_S \notin F(N,X)_S.$$

Proof: Let

$$\delta < \left(2^r (2C_L)^{2D}\right)^{-2/\epsilon}$$

be such that

$$\frac{S(X,\delta,d)}{|\log \delta|} \leq \mathrm{mdim}_{\mathrm{M}}(X,d) + \epsilon/4.$$

If N is large enough (depending on ϵ and δ), X can be covered by $\delta^{-(D+\epsilon/2)N}$ balls of the form

$$B_X(x,N,\delta) = \{x' \in X \colon d(T^k x, T^k x') < \delta \text{ for all } 0 \leq k < N\}.$$

Since C_L is a Lipschitz constant for all w_i

$$F(N, B_X(x,N,\delta)) \subset \{a \in [0,1]^{rN} \colon \|F(N,x) - a\|_\infty < C_L\delta\},$$

the set $F(N,X)$ can be covered by $\delta^{-(D+\epsilon/2)N}$ balls in the $\|\cdot\|_\infty$ norm of radius $C_L\delta$. Enumerate these balls as $B(k)$, $k = 1, \ldots, K = \delta^{-(D+\epsilon/2)N}$.

Choose ξ with uniform probability in $[0,1]^{rN}$. For any S

$$P(\xi_S \in F(N,X)_S) \leq \sum_{k=1}^{K} P(\xi_S \in B(k)|_S) \leq \delta^{-(D+\epsilon/2)N}(2C_L\delta)^{|S|}.$$

Hence

$$\begin{aligned} P\big(\exists S \colon |S| \geq (D+\epsilon)N \text{ and } \xi_S \in F(N,X)_S\big) \leq \\ &\leq \sum_{|S|\geq(D+\epsilon)N} P(\xi_S \in F(N,K)_S) \\ &\leq (\#\text{ of such } S) \times \delta^{-(D+\epsilon/2)N}(2C_L\delta)^{(D+\epsilon)N} \\ &\leq 2^{rN}((2C_L)^{2D}\delta^{\epsilon/2})^N \ll 1, \end{aligned}$$

and so, with high probability, a random ξ will satisfy the requirements. ∎

LEMMA 4.4: *If $\pi \colon F(N,X) \to [0,1]^{rN}$ satisfies for both $a = 0$ and 1, and all $\xi \in [0,1]^{rN}$,*

$$\{1 \leq k \leq rN \colon \xi_k = a\} \subset \{1 \leq k \leq rN \colon \pi(\xi)|_k = a\},$$

then $\pi \circ F(N,\cdot)$ is compatible with α_0^{N-1}

Proof: Indeed, if $\xi \in [0,1]^{rn}$, define for $0 \leq j < N$ and $1 \leq i < r$,

$$W_{i,j} = \begin{cases} T^{-j}U_i & \text{if } \xi_{jr+i} = 0, \\ T^{-j}V_i & \text{otherwise.} \end{cases}$$

It is easy to see that

$$\pi^{-1}(\xi) \subset \bigcap_{\substack{1\leq i<r \\ 0\leq j<N}} W_{i,j} \in \alpha_0^{N-1}. \qquad ∎$$

Proof of Theorem 4.2: Let $\epsilon > 0$. Find $\bar{\xi}$, N as in Lemma 4.3. Let

$$\Phi = \big\{\xi \in [0,1]^{rN} \colon \xi_k = \bar{\xi}_k \text{ for more than } (D+\epsilon)N \text{ indexes } k\big\};$$

thus $F(N,X) \subset [0,1]^{rN} \setminus \Phi$. For brevity we shall denote the latter set as Φ^C. We shall construct a continuous retraction π of Φ^C onto the $\lfloor (D+\epsilon)N \rfloor + 1$ skeleton of the cube $I = [0,1]^{rN}$. For $m = 1, 2, \dots$, we denote by J_m the set

$$J_m = \{\xi \in I\colon \xi_i \in \{0,1\} \text{ for at least } m \text{ indices } 1 \le i \le rN\}.$$

Since $\bar{\xi}$ is in the interior of I, one can define $\pi_1\colon I \setminus \{\bar{\xi}\} \to J_1$ by mapping each ξ to the intersection of the ray starting at $\bar{\xi}$ and passing through ξ and J_1. For each of the $(rN-1)$-dimensional cubes I^l that comprise J_1, we can define a retraction on I^l in a similar fashion using as a center the projection of $\bar{\xi}$ onto I^l. This will define a continuous retraction π_2 of Φ^C into J_2. As long as there is some intersection of Φ with the cubes in J_m this process can be continued, thus we get finally a continuous projection π of Φ^C onto J_{m_0}, with

$$m_0 + \lfloor (D+\epsilon)N \rfloor + 1 = n.$$

Clearly, π satisfies the conditions of Lemma 4.4. Thus $\pi \circ F(N,\cdot) \succ \alpha_0^{N-1}$. Also, as $F(N,\cdot) \subset \Phi^C$,

$$\pi \circ F(N,\cdot) \subset J_{m_0},$$

the latter having topological dimension $\lfloor (D+\epsilon)N \rfloor + 1$.

To summarize: we have constructed an α_0^{N-1} compatible function from X to a space of topological dimension $\le (D+\epsilon)N + 1$, and so

$$\mathcal{D}(\alpha_0^{N-1}) \le (D+\epsilon)N + 1.$$

As $\epsilon > 0$ can be made as small as we like, α as refined as we like, we see that indeed

$$\operatorname{mdim}(X) \le D. \quad \blacksquare$$

It is plausible that equality actually holds in Theorem 4.2. In general this is still open, but Lindenstrauss [7] contains a proof of this for $\mathbb{Z}$-actions, if in addition (X,T) is an extension of a minimal system.

5. The small-boundary property

Definition 5.1: Let (X,T) be a dynamical system, and E a subset of X. We define the **orbit capacity** of the set E to be

$$\operatorname{ocap}(E) = \lim_{n\to\infty} \sup_{x\in X} \frac{\sum_{i=0}^{n-1} 1_E(T^i x)}{n}.$$

A set $E \subset X$ will be called **small** if $\operatorname{ocap}(E) = 0$.

We remark that the limit above exists since

$$\sup_{x\in X} \sum_{i=0}^{n+m-1} 1_E(T^i x) \le \sup_{x\in X} \sum_{i=0}^{n-1} 1_E(T^i x) + \sup_{x\in X} \sum_{i=0}^{m-1} 1_E(T^i x).$$

Definition 5.2: A dynamical system (X,T) has the **small-boundary property** (SBP) if every point $x \in X$ and every open $U \ni x$ there is a neighborhood $V \subset U$ of x with small boundary.

The notion of SBP can be thought of as an analogue of the usual definition of zero dimensional space: X is zero dimensional if for every point $x \in X$ and every open $U \ni x$ there is a neighborhood $V \subset U$ of x with empty boundary.

The definition of small sets was introduced in Shub and Weiss [11], where it was shown that if X is uniquely ergodic (or even if it has less than $2^{\aleph_0}$ ergodic T-invariant measures) then X has the SBP. Indeed, first we note that for any closed set E that is not small there is some invariant probability measure μ such that $\mu(E) > 0$ — let x_k be a sequence such that

$$\frac{1}{N_k} \sum_{i=0}^{N_k-1} 1_E(T^i x_k) \ge c > 0;$$

then any weak* limit point of

$$\frac{1}{N_k} \sum_{i=0}^{N_k-1} \delta_{T^i x_k}$$

is invariant and satisfies $\mu(E) \ge c$. Conversely, if E is small and μ an invariant measure

$$\mu(E) = \frac{1}{n} \int \sum_{i=0}^{n-1} 1_E(T^i x) d\mu(x) \to 0.$$

Now, if X is uniquely ergodic, and μ the unique invariant measure, then for any $x \in X$, the set

$$\big\{r \in \mathbb{R}^+\colon \{y\colon d(x,y) = r\} \text{ is not small}\big\} = \{r \in \mathbb{R}^+\colon \mu\{y\colon d(x,y) = r\} \ne 0\}$$

is countable, and so there are many arbitrarily small balls around x with small boundary, and X has the SBP.

The following proposition is an immediate consequence of the definitions:

PROPOSITION 5.3: *If (X,T) has the SBP, then for every open cover α of X and every ϵ there is a subordinate partition of unity $\phi_j\colon X \to [0,1]$ $(j = 0 \ldots |\alpha|)$ such that*

$$\sum_{j=1}^{|\alpha|} \phi_j(x) \equiv 1,$$

$$\mathrm{supp}(\phi_j) \subset U \quad \text{for some } U \in \alpha, \text{ and } j = 1, \ldots, |\alpha|,$$

$$\mathrm{ocap}\Big(\bigcup_{j=1}^{|\alpha|} \phi_j^{-1}(0,1)\Big) < \epsilon. \tag{5.1}$$

Proof: First find (using the SBP) a cover of X by open sets with small boundary that refines α, and by taking unions of these sets it is possible to find a refinement $\alpha' \succ \alpha$ such that there is a one-to-one correspondence between the elements U_j of α and U'_j of α' with $U'_j \subset U_j$.

Let N satisfy

$$\frac{1}{N}\sum_{i=0}^{N-1} 1_{\mathrm{bd}\,U'_j}(T^i x) < \frac{\epsilon}{|\alpha|}, \quad \text{for all } x \in X \text{ and } j,$$

and δ be small enough so that for all j

$$\frac{1}{N}\sum_{i=0}^{N-1} 1_{B(\mathrm{bd}\,U'_j,\delta)}(T^i x) < \frac{\epsilon}{|\alpha|},$$

where $B(\mathrm{bd}\,U'_j,\delta)$ is the set of all points of distance smaller than δ from $\mathrm{bd}\,U'_j$. We also take δ to be small enough so that $B(\mathrm{bd}\,U'_j,\delta) \subset U_j$ for all j. Now take

$$\psi_j(x) = \begin{cases} 1 & \text{if } x \in U'_j, \\ \max(0, 1 - \delta^{-1} d(x, \mathrm{bd}\,U'_j)) & \text{otherwise.} \end{cases}$$

We define the functions $\phi_i(x)$ as follows:

$$\begin{aligned} \phi_1(x) &= \psi_1(x), \\ \phi_2(x) &= \min(\psi_2(x), 1 - \phi_1(x)), \\ \phi_3(x) &= \min(\psi_3(x), 1 - \phi_1(x) - \phi_2(x)), \\ &\vdots \end{aligned}$$

These clearly satisfy the required conditions. ∎

THEOREM 5.4: *Every space with the SBP has mean dimension zero.*

Proof: Let (X,T) have the SBP, α a cover of X. Take $\epsilon > 0$. Construct, according to the above proposition, an α subordinate partition of unity which obeys (5.1). Let $|\alpha| = k$, $A = \bigcup_{i=1}^{k} \phi_j^{-1}(0,1)$. Assume N is large enough so that

$$\frac{1}{N}\sum_{i=0}^{N-1} 1_A(T^{-i}x) < \epsilon \quad \text{for all } x \in X. \tag{5.2}$$

Define $\Phi\colon (X,T) \to \mathbb{R}^k$ by

$$x \mapsto (\phi_1(x), \dots, \phi_k(x)).$$

Define the map $f_N\colon X \to \mathbb{R}^{kN}$ by

$$f_N(x) = (\Phi(x), \Phi(Tx), \dots, \Phi(T^{N-1}x)).$$

We claim that $f_N(X)$ is a subset of a finite number of ϵkN dimensional affine subspaces of $\mathbb{R}^{kN}$.

Indeed, let e_j^i, $i = 1, \dots, N$, $j = 1, \dots, k$ be the standard base of $\mathbb{R}^{kN}$. Define for every $I = \{i_1, i_2, \dots, i_{N'}\}$, $N' < \epsilon N$, and every $\xi \in \{0,1\}^{kN}$

$$C(I,\xi) = \operatorname{span}\{e_j^i\colon i \in I,\ 1 \le j \le k\} + \xi.$$

Then by (5.2),

$$f_N(X) \subset \bigcup_{|I|<\epsilon N,\, \xi} C(I,\xi).$$

It is easy to see that f_N is α_0^{N-1}-compatible. By Proposition 2.4 we see that

$$\mathcal{D}(\alpha_0^{N-1}) < \epsilon kN$$

and so (X,T) has zero mean dimension. ∎

Note that in general the converse to this theorem may be false. Indeed, consider the dynamical system $X = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ and $T\colon (x,y) \mapsto (x, y + x \bmod 1)$. X is finite dimensional, hence $\operatorname{mdim} X = 0$, but no small neighborhood of any point of the form $(0,y)$ has small boundary, since this boundary must contain at least two points fixed by T. There are, however, cases where the converse holds (Lindenstrauss [7]), and presumably a better definition of the SBP that takes into account the periodic points should be equivalent to $\operatorname{mdim} X = 0$. See Lindenstrauss [6] for a modification of the SBP that holds for finite dimensional dynamical systems.

6. Appendix: Subadditive functions on amenable groups

Let G be a discrete (countable) amenable group. According to Følner's characterization of such groups for any finite $K \subset G$ and any $\delta > 0$ there is a finite set F that is (K, δ)-invariant in the sense that

$$|\{f \in F\colon Kf \subset F\}| \geq (1-\delta)|F|.$$

If ϕ is a nonnegative monotone subadditive function defined on the finite subsets of G that is right translation invariant, i.e.

$$\phi(Ag) = \phi(A) \qquad \text{for all } g \in G,\ A \subset G,$$

then as sets F become more and more invariant $\phi(A)/|A|$ converges to a limit. More precisely, there is a finite limit b such that for all $\epsilon > 0$, there is a finite $K \subset G$ and $\delta > 0$ such that for any F that is (K, δ) invariant we have

$$|\phi(F)/|F| - b| < \epsilon.$$

The classical application of this fact is in defining the entropy of a G-indexed stationary process (Ornstein and Weiss [9]). For the record we present here a proof of the general fact.

THEOREM 6.1: *If G is amenable and ϕ is defined on the finite subsets of G and satisfies:*

1. $0 \leq \phi(A) < +\infty$ *and* $\phi(\emptyset) = 0$,
2. $\phi(A) \leq \phi(B)$ *for all* $A \subset B$,
3. $\phi(Ag) = \phi(A)$ *for all* $A \subset G$ *and* $g \in G$,
4. $\phi(A \cup B) \leq \phi(A) + \phi(B)$ *if* $A \cap B = \emptyset$,

then $\frac{1}{|A|}\phi(A)$ *converges to a limit as the set A becomes more and more invariant.*

It is worth pointing out that there is some novelty in the theorem even in the case $G = \mathbb{Z}$. A special case of the theorem for $G = \mathbb{Z}$ is a well known calculus exercise — proving that for any subadditive sequence $0 \leq a_n$ (i.e. one satisfying: $a_{n+m} \leq a_n + a_m$) the limit $\lim \frac{a_n}{n}$ exists.

Since the proof of the theorem is based on the proof of this exercise let us briefly recall it. Set

$$z = \inf \frac{a_n}{n}.$$

If $\epsilon > 0$ is given and N chosen so that $\frac{a_N}{N} \leq z + \epsilon$, then for all $n > N$, writing $n = qN + r$ with $r < N$ we see that

$$a_n \leq a_{qN} + a_r \leq q a_N + a_r,$$

which upon division by n gives

$$\frac{a_n}{n} \leq \frac{qN}{n} \cdot \frac{a_N}{N} + \frac{a_r}{n} \leq \frac{qN}{n}(z+\epsilon) + \frac{a_r}{n},$$

so that $\overline{\lim} \frac{a_n}{n} \leq z + \epsilon$, which shows that $\lim \frac{a_n}{n}$ equals z.

Such a sequence corresponds to a translation invariant subadditive function defined not on all finite subsets of $\mathbb{Z}$ but only on intervals. The key property of intervals used in the above proof is that any interval tiles, via disjoint translations, any large interval, up to a small error. For general finite sets in $\mathbb{Z}$ this is no longer true, so that taking an *infimum* over all finite sets does not appear to work. What is needed is a substitute for this tiling property, and we shall use the machinery of quasi-tiling by approximately invariant sets, as developed in Ornstein and Weiss [9]. Here are the notions that are necessary to formulate the results that we need:

Definition 6.2: A collection of finite sets $\{E_j\}$ is said to be ϵ-disjoint if there are subsets $\hat{E}_j \subset E_j$ satisfying:

- $\hat{E}_j \cap \hat{E}_i = \emptyset$ for all $j \neq i$,
- $|\hat{E}_j|/|E_j| \geq 1 - \epsilon$ for all j.

Definition 6.3: A collection of sets $\{T_i\colon 1 \leq i \leq M\}$ in a group G is said to **ϵ-quasi tile** the group if there is a finite $K \subset G$, $\eta > 0$, and translates $\{T_i c_{ij}\}_{i,j}$ of the T_i that are ϵ-disjoint, such that for any F that is (K, η)-invariant one has

$$|F \cap \bigcup_{ij} T_i c_{ij}|/|F| \geq 1 - \epsilon.$$

The following theorem is in Ornstein and Weiss [9], I.2:

THEOREM 6.4: *For any amenable group G, given $\epsilon > 0$, there is an M that depends only on ϵ, such that if*

$$e \in T_1 \subset T_2 \subset \cdots \subset T_M \subset G$$

satisfy that T_{i+1} is $(T_i T_i^{-1}, \eta_i)$-invariant, $i = 1, 2, \ldots M-1$ (η_i depends on i and T_i), then the $\{T_i\colon 1 \leq i \leq M\}$ ϵ-quasi tile G.

The formal meaning of the fact that η_i depends on i and T_i is that there is a function Ψ from $\mathbb{N} \times \{\text{finite subsets of } G\}$ to $\mathbb{R}^+$, such that if $\eta_i \leq \Psi(i, T_i)$ for $i = 1, 2, \ldots, M-1$ then the conclusion of the theorem is valid.

Let K_n be an increasing sequence of finite sets whose union is G and set

$$\mathcal{F}_n = \{F \subset G\colon F \text{ is finite and } (K_n, 1/n) \text{ invariant}\}$$

(note that $\mathcal{F}_1 \supset \mathcal{F}_2 \supset \cdots$). If $\phi(A)$ is a translation invariant subadditive function defined on the finite subsets of G, set

$$a_n = \inf_{F \in \mathcal{F}_n} \phi(F)/|F|$$

and put $a_0 = \lim_{n\to\infty} a_n$. We shall show that for any $\delta > 0$, there is some $n(\delta)$ such that

$$|\phi(F)/|F| - a_0| < \delta \qquad \text{for all } F \in \mathcal{F}_{n(\delta)}. \tag{6.1}$$

To see this, fix some $\epsilon > 0$ (which will be chosen later to be small, in a fashion that depends only on δ), and then let N be large enough for the conclusion of Theorem 6.4 to hold for this ϵ.

Inductively, find sets $\{F_1, \ldots, F_N\}$ that both quasi tile and also satisfy

$$\phi(F_i)/F_i \leq a_0 + \delta/2, \qquad 1 \leq i \leq N. \tag{6.2}$$

Let $n(\delta)$ be large enough so that any $F \in \mathcal{F}_{n(\delta)}$ is itself ϵ-quasi tiled by $\{F_1, \ldots, F_N\}$. We can now bound $\phi(F)$ by using (6.2) and the fact that $\{F_i\}$ ϵ-quasi tile:

$$\phi(F)/F \leq (a_0 + \delta/2)(1+\epsilon) + \epsilon\phi(\{e\}),$$

and so if ϵ is chosen to be small enough we get (6.1), thus proving Theorem 6.1. ∎

We note that if ϕ satisfies a stronger subadditive inequality — namely

$$\phi(A \cup B) + \phi(A \cap B) \leq \phi(A) + \phi(B)$$

for all A, B disjoint or not, then an easier proof can be given which does not require the ϵ-quasi tiling machinery. Such a result can be found in Kieffer [12], Emerson [13], or in Ollagnier [8], section 2.2. The measure theoretic entropy of the span of a partition under a measure preserving action satisfies this stronger inequality, as a consequence of the properties of *conditional entropy*, for which there is no clear analogue in the context of mean dimension.

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